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Rational New Approach to the Response of an Aircraft Encountering Atmospheric Turbulence

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Based on the technique of expanding a random function in terms of statistically orthogonal random functionals, a method is developed for analyzing the response of an aircraft to a stationary, weakly non-Gaussian atmospheric turbulence. The atmospheric turbulence is characterized through the correlation functions, and no other ad hoc assumptions are necessary. The expansion kernels enable us to construct various probability density functions of interest—e.g., the joint probability density function of the response velocity and acceleration. To illustrate the method and demonstrate some of the effects due to the non-Gaussian nature of atmospheric turbulence, the simple case of the plunging motion of a rigid aircraft in a turbulent gust is presented in detail. It is found that even when the aircraft responds as a linear system, a nearly Gaussian turbulent gust can render an aircraft response highly non-Gaussian.

I. Introduction

In the gust loading problem, certain calculations (such as the level crossing frequency) require explicit knowledge of the probability density functions of the aircraft motion in response to atmospheric turbulence. Traditionally, these probability density functions are obtained by assuming that the turbulence is Gaussian and that the aircraft responds as a linear system. By these two assumptions, it follows that the probability structure of the aircraft response is also Gaussian. Recent observations, however, have revealed that the assumption of a Gaussian turbulence is not satisfactory since it significantly underestimates the frequency of occurrence of high gust velocities and, consequently, gust loading.[1]

To amend this shortcoming, a number of new atmospheric turbulence models have recently been proposed[1-3] for the aircraft gust interaction problem. In Ref. 1, the turbulent gust is assumed to be the product of a deterministic function of time and a stationary Gaussian random process. The resulting modulated Gaussian process is nonstationary. The rationale of the assumed modulating function is not given. In Ref. 2, the turbulent gust is modeled as the sum of two components: a stationary Gaussian process and a modulated Gaussian process. The modulating function itself is a stationary Gaussian process. The three Gaussian random functions are assumed to be mutually independent. The two components have the same spectrum. In Ref. 3, it is suggested that the two components in Ref. 2 should have different spectra and that the modulating function is not Gaussian.

Instead of making physical models, we will approach the problem by using a mathematically well-developed technique for dealing with a non-Gaussian process. The techniques of expanding a given random function in terms of statistically orthogonal random functionals is not unfamiliar in the field of radio communication engineering. These techniques are essentially originated by Wiener[4] and further developed by Kuznetsov et al.[5] The last two lectures of Ref. 4 are devoted to a discussion of possible application of these techniques to the problems of hydrodynamic turbulence. An application of these techniques to the isotropic turbulence is presented in Ref. 6. These techniques are mathematically rigorous.

In this paper, we present the method developed based on these techniques for analyzing the response of an aircraft flying through a non-Gaussian turbulence. To the best knowledge of the present authors, their application to the gust loading problem represents a new approach. In Section II, a brief introduction of the theoretical background is given. Section II also presents a novel procedure for obtaining the probability density functions of a stationary random function represented by two-term expansion.

In Section III, these techniques are applied to analyze the plunging motion of a rigid aircraft as induced by the turbulent gust.[1,7] It is demonstrated there that, even though the aircraft responds as a linear system, a nearly Gaussian turbulent gust can render an aircraft response highly non-Gaussian.

II. Theoretical Background

A. Functional Expansion of a Random Function

By Wiener's technique,[4] any given random function $Y(t)$ can be expanded in terms of the polynomial of the Gaussian white noise function $a(t)$, which is a normally distributed random function with the first and second moments given by

$$\langle a(t) \rangle = 0 \tag{1a}$$

$$\langle a(t)a(t') \rangle = \delta(t-t') \tag{1b}$$

where $\langle \cdot \rangle$ stands for ensemble average. The expansion is written as

$$Y(t) = \sum_{n=0}^{\infty} \int \cdots \int T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n) \times H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots\mathrm{d}\tau_n \tag{2}$$

where $T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)$ are deterministic functions of $t,\tau_1,\tau_2,\cdots\tau_n$, and $H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)$ are Hermite polynomials of $a(\tau)$[8]:

$$H^{(0)} = 1 \tag{3a}$$

$$H^{(1)}(\tau) = a(\tau) \tag{3b}$$

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$$H^{(2)}(\tau_1,\tau_2)=a(\tau_1)a(\tau_2)-\delta(\tau_1-\tau_2) \tag{3c}$$

The domain of integration is taken from $-\infty$ to $+\infty$. Since $H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)$ are symmetric in $\tau_1,\tau_2,\cdots\tau_n$, so are $T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)$.

If $Y(t)$ is a stationary random function with zero mean, then the expansion takes the form

$$Y(t)=\sum_{n=1}^{\infty}\int\cdots\int T^{(n)}(t-\tau_1,t-\tau_2,\cdots t-\tau_n)\times H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots\mathrm{d}\tau_n \tag{4}$$

and $T^{(n)}(t-\tau_1,t-\tau_2,\cdots t-\tau_n)$ are symmetric in $t-\tau_1,t-\tau_2,\cdots t-\tau_n$.

It has been shown by Cameron and Martin[9] that the expansion (2) is complete. It is noted that the first term of the expansion (4) is Gaussian. The succeeding terms represent the departure of $Y(t)$ from Gaussianity. Further mathematical details about the expansion and its application to the problems of homogeneous isotropic turbulence can be found in Ref. 6 and the references cited there. An attempt to apply this expansion to describe the statistical properties of atmospheric turbulence has been previously made by Dutton.[7]

If the statistical properties of $Y(t)$ are given, two methods have been proposed to find the kernels in the expansion. The first method[4,5] is based on the criterion of mean square convergence, equivalent to minimizing the error,

$$\epsilon=Y(t)-\sum_{n=0}^{\infty}\int\cdots\int T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)\times H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots\mathrm{d}\tau_n \tag{5}$$

Under the criterion of minimizing the mean square error ϵ, it is proved[5] that the kernel functions are defined by the following integral equation:

$$\langle Y(t)H^{(n)}(\tau_1',\tau_2',\cdots\tau_n')\rangle=\int\cdots\int T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)\langle H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)H^{(n)}(\tau_1',\tau_2',\cdots\tau_n')\rangle\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots\mathrm{d}\tau_n \tag{6}$$

However, the joint statistical properties of $Y(t)$ and $H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)$ are in general not known; further considerations for the evaluation of the quantities defined in the left-hand side of Eq. (6) are needed. Once the left-hand side is given, $T^{(n)}(t;\tau_1,\tau_2,\cdots\tau_n)$ can be found because the correlation of $H^{(n)}(\tau_1,\tau_2,\cdots\tau_n)$ and $H^{(n)}(\tau_1',\tau_2',\cdots\tau_n')$ is a series of delta functions.

For cases where $Y(t)$ is stationary and nearly Gaussian, the expansion (4) may be truncated after the second term[6,7]:

$$Y(t)=Y_1(t)+Y_2(t) \tag{7a}$$

$$Y_1(t)=\int_{-\infty}^{\infty}T^{(1)}(t-\tau)H^{(1)}(\tau)\mathrm{d}\tau \tag{7b}$$

$$Y_2(t)=\int_{-\infty}^{\infty}T^{(2)}(t-\tau_1,t-\tau_2)H^{(2)}(\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \tag{7c}$$

Here, $Y_1(t)$ is Gaussian; $Y_2(t)$ is a small correction due to $Y(t)$ being non-Gaussian. Expressing an atmospheric turbulence velocity in the form of Eq. (7), Dutton, in Ref. 7, adopted the method mentioned above and evaluated the left-hand side of Eq. (6) by digital simulation. The kernels $T^{(1)}$ and $T^{(2)}$ were then solved by Fourier-transform techniques, but with "formidable difficulties." Nevertheless, he obtained enough results to show a comparison of part of an input spectrum and the reconstructed version from his results of the expansion technique. The data processing aspects seemed to be discouraging, and his project was discontinued.

The second method, also presented in Ref. 5, is based on the equilibration of the correlation functions of both sides of Eq. (4). If the correlation functions have been determined, the expansion kernels $T^{(n)}$ are governed by an infinite set of simultaneous nonlinear integral equations. We have followed this route in the present study. For the two-term expansion (7), our example in Section III shows that the solution is straightforward via an iterative procedure.

B. Probability Density Functions

Even if $T^{(1)}$, $T^{(2)}$, etc., are assumed known, we can find no previous work on how to construct the various probability density functions of interest. It is therefore necessary first to develop a method to do so. Let us consider the joint probability density function, $p(y,\dot y)$, where y and $\dot y$ are, respectively, the real values of the stationary random functions $Y(t)$ and its time derivative $\dot Y(t)$ at any instant t. When $Y(t)$ is described by Eq. (7),

$$\dot Y(t)\equiv\frac{\mathrm{d}}{\mathrm{d}t}Y(t)=\int_{-\infty}^{\infty}\dot T^{(1)}(t-\tau)H^{(1)}(\tau)\mathrm{d}\tau+\int_{-\infty}^{\infty}\int\dot T^{(2)}(t-\tau_1,t-\tau_2)H^{(2)}(\tau_1,\tau_2)\mathrm{d}\tau_1,\mathrm{d}\tau_2 \tag{8}$$

where

$$\dot T^{(1)}(t-\tau)=\frac{\partial}{\partial t}T^{(1)}(t-\tau)=\frac{\mathrm{d}T^{(1)}}{\mathrm{d}\eta} \tag{9a}$$

$$\dot T^{(2)}(t-\tau_1,t-\tau_2)=\frac{\partial}{\partial\eta_1}T^{(2)}(\eta_1,\eta_2)+\frac{\partial}{\partial\eta_2}T^{(2)}(\eta_1,\eta_2) \tag{9b}$$

η,η_1, and η_2 are, respectively, abbreviations for $t-\tau$, $t-\tau_1$, and $t-\tau_2$. As we are interested in a physically realizable system, $T^{(1)}(\eta)$ and $T^{(2)}(\eta_1,\eta_2)$ will have the value zero whenever any of the arguments is negative.[10] Hence, the domains of definition for $T^{(1)}(\eta)$ and $T^{(2)}(\eta_1,\eta_2)$ are $0\le\eta<\infty$ and $0\le\eta_1<\infty$, $0\le\eta_2<\infty$, respectively.

A natural complete set of eigenfunctions in $[0,\infty)$ is the Laguerre function[11]:

$$\phi_m(\eta)=\frac{L_m(\eta)}{m!}e^{-\eta/2} \tag{10a}$$

where $L_m(\eta)$ is the Laguerre polynomial defined as

$$L_m(\eta)=e^{\eta}\frac{\mathrm{d}^m}{\mathrm{d}\eta^m}(\eta^m e^{-\eta}) \tag{10b}$$

The expansions of $T^{(1)}(\eta)$, $T^{(2)}(\eta_1,\eta_2)$, $\dot T(\eta)$, and $\dot T^{(2)}(\eta_1,\eta_2)$ in the series of $\phi_m(\eta)$ may be written as

$$T^{(1)}(\eta)=\sum_{m=0}^{N}D_m\phi_m(\eta) \tag{11}$$

$$\dot T^{(1)}(\eta)=\sum_{m=0}^{N}\bar D_m\phi_m(\eta) \tag{12}$$

$$T^{(2)}(\eta_1,\eta_2)=\sum_{m=0}^{N}\sum_{n=0}^{N}A_{mn}\phi_m(\eta_1)\phi_n(\eta_2) \tag{13}$$

and

$$\dot T^{(2)}(\eta_1,\eta_2)=\sum_{m=0}^{N}\sum_{n=0}^{N}\bar A_{mn}\phi_m(\eta_1)\phi_n(\eta_2) \tag{14}$$

where $N \to \infty$. The expansion coefficients D_m, $\bar{D}_m$, A_{mn}, and $\bar{A}_{mn}$ are determined by

$$D_m = \int_0^\infty T^{(1)}(\eta)\phi_m(\eta)\,d\eta \tag{15}$$

$$\bar{D}_m = \int_0^\infty \dot{T}^{(1)}(\eta)\phi_m(\eta)\,d\eta \tag{16}$$

$$A_{mn} = \int_0^\infty \int T^{(2)}(\eta_1,\eta_2)\phi_m(\eta_1)\phi_n(\eta_2)\,d\eta_1 d\eta_2 \tag{17}$$

and

$$\bar{A}_{mn} = \int_0^\infty \int \dot{T}^{(2)}(\eta_1,\eta_2)\phi_m(\eta_1)\phi_n(\eta_2)\,d\eta_1 d\eta_2 \tag{18}$$

Substituting Eqs. (11-14) into Eqs. (7) and (8), we obtain

$$Y(t) = \int_{-\infty}^{t} \sum_{m=0}^{N} D_m\phi_m(t-\tau)H^{(1)}(\tau)\,d\tau + \int_{-\infty}^{\infty}\int \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}\phi_m(t-\tau_1)\phi_n(t-\tau_2) \times H^{(2)}(\tau_1,\tau_2)\,d\tau_1 d\tau_2 \tag{19}$$

$$\dot{Y}(t) = \int_{-\infty}^{t} \sum_{m=0}^{N} \bar{D}_m\phi_m(t-\tau)H^{(1)}(\tau)\,d\tau + \int_{-\infty}^{t}\int \sum_{m=0}^{N}\sum_{n=0}^{N} \bar{A}_{mn}\phi_m(t-\tau_1)\phi_n(t-\tau_2) \times H^{(2)}(\tau_1,\tau_2)\,d\tau_1 d\tau_2 \tag{20}$$

It is assumed that the behaviors of $T^{(1)}(\eta)$, $\dot{T}^{(2)}(\eta_1,\eta_2)$, and $\dot{Y}^{(2)}(\eta_1,\eta_2)$ are nice enough so that the series expansions, Eqs. (11-14), converge uniformly in their domains of definition. We may therefore[12] interchange the order of summation and integration in these two equations. From Eq. (19),

$$Y(t) = \sum_{m=0}^{N} D_m \int_{-\infty}^{t} \phi_m(t-\tau)H^{(1)}(\tau)\,d\tau + \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}\int_{-\infty}^{t}\int \phi_m(t-\tau_1)\phi_n(t-\tau_2) \times H^{(2)}(\tau_1,\tau_2)\,d\tau_1 d\tau_2 \tag{21}$$

Defining

$$X_m(t) = \int_{-\infty}^{t} \phi_m(t-\tau)a(\tau)\,d\tau \tag{22}$$

and using Eqs. (3b) and (3c) for $H^{(1)}(\tau)$ and $H^{(2)}(\tau_1,\tau_2)$, we obtain

$$Y(t) = \sum_{m=0}^{N} D_m X_m(t) + \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}X_m(t)X_n(t) - \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}\int_{-\infty}^{t}\int_{-\infty}^{t} \phi_m(t-\tau_1)\phi_n(t-\tau_2) \times \delta(\tau_1-\tau_2)\,d\tau_1 d\tau_2 \tag{23}$$

The last term of Eq. (23) can be simplified by using the orthonormality property of $\phi_m(\eta)$. The result is

$$Y(t) = \sum_{m=0}^{N} D_m X_m(t) + \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}X_m(t)X_n(t) - \sigma_1 \tag{24}$$

where

$$\sigma_1 \equiv \sum_{m=0}^{N} A_{mm}$$

Similarly, we obtain from Eq. (20):

$$\dot{Y}(t) = \sum_{m=0}^{N} \bar{D}_m X_m(t) + \sum_{m=0}^{N}\sum_{n=0}^{N} \bar{A}_{mn}X_m(t)X_m(t) - \bar{\sigma}_1 \tag{25}$$

where

$$\bar{\sigma}_1 = \sum_{m=0}^{N} \bar{A}_{mm}$$

It is noted that $X_m(t)$ defined by Eq. (22) has the following properties:

1) $X_m(t)$ is a Gaussian random function with zero mean because $a(t)$ is Gaussian with zero mean.

2) $X_m(t)$ is stationary because the argument of the kernel is the difference of the independent variables.

3) $X_m(t)$ and $X_n(t)$ are statistically independent because $X_m(t)$ and $X_n(t)$ are both stationary Gaussian and their covariance is

$$\langle X_m(t)X_n(t)\rangle = \int_{-\infty}^{t}\int_{-\infty}^{t} \phi_m(t-\tau_1)\phi_n(t-\tau_2) \times \langle a(\tau_1)a(\tau_2)\rangle\,d\tau_1 d\tau_2, \quad =0 \quad \text{if } m\neq n \tag{26a}$$

4) $X_m(t)$ has unit variance because

$$\langle X_m^2\rangle = \int_0^\infty \phi_n(\eta)\phi_m(\eta)\,d\eta = 1 \tag{26b}$$

Based on these properties for X_0, $X_1,\ldots X_n$, their joint probability density function is

$$p(x_0,x_1,\cdots x_n) = (2\pi)^{-(N+1)/2}\cdot\prod_{m=0}^{N}\exp\left(-\frac{x_m^2}{2}\right) \tag{27}$$

To find the joint probability density function $p(y,\dot{y})$ of the random functions $Y(t)$ and $\dot{Y}(t)$, we proceed from the characteristic function.[13] The characteristic function $\phi_{Y\dot{Y}}(t_1,t_2)$ is defined as the Fourier pair of $p(y,\dot{y})$:

$$p(y,\dot{y}) = (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{Y\dot{Y}}(t_1,t_2)e^{-i(t_1y+t_2\dot{y})}\,dt_1 dt_2$$

$$\phi_{Y\dot{Y}}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(y,\dot{y})e^{i(t_1y+t_2\dot{y})}\,dy d\dot{y}$$

Hence, with Eqs. (24) and (25),

$$\phi_{Y\dot{Y}}(t_1,t_2) = \int\cdots\int \exp\Big\{it_1\Big[\sum_{m=0}^{N} D_m x_m + \sum_{m=0}^{N}\sum_{n=0}^{N} A_{mn}x_m x_n - \sigma_1\Big] + it_2\Big[\sum_{m=0}^{N} \bar{D}_m x_m + \sum_{m=0}^{N}\sum_{n=0}^{N} \bar{A}_{mn}x_m x_n - \bar{\sigma}_1\Big]\Big\} \times p(x_0,x_1,\cdots x_n)\,dx_0 dx_1\cdots dx_n \tag{28}$$

Let us rewrite Eq. (28) in the following form:

$$\phi_{Y\dot{Y}}(t_1,t_2) = \int\cdots\int \exp\Big\{i\Big[\sum_{m=0}^{N} P_m(t_1,t_2)x_m + \sum_{m=0}^{N}\sum_{n=0}^{N} \phi_{mn}(t_1,t_2)x_m x_n - (\sigma_1 t_1 + \bar{\sigma}_1 t_2)\Big]\Big\} \times p(x_0,x_1,\cdots x_n)\,dx_0 dx_1\cdots dx_N \tag{29}$$

where

$$P_m(t_1,t_2) \equiv D_m t_1 + \bar{D}_m t_2 \tag{30}$$

and

$$Q_{mn}(t_1,t_2) \equiv A_{mn} t_1 + \bar{A}_{mn} t_2 \tag{31}$$

Since A_{mn} and $\bar{A}_{mn}$ are symmetric with respect to their indices, so is $Q_{mn}(t_1,t_2)$. For simplicity, when there is no danger of confusion, we will write P_m for $P_m(t_1,t_2)$ and Q_{mn} for $Q_{mn}(t_1,t_2)$. The matrix $\bar{\bar{Q}} = \{Q_{mn}\}$ is real and symmetric. Some important properties related to the matrix $\bar{\bar{Q}}$ may be noted:

1) The characteristic roots of $\bar{\bar{Q}}$ are all real.
2) There exists a real orthogonal matrix $\bar{\bar{B}}$ such that

$$\bar{\bar{B}}^{-1}\bar{\bar{Q}}\bar{\bar{B}} = \bar{\bar{B}}^T\bar{\bar{Q}}\bar{\bar{B}} = \operatorname{diag}\{\lambda_0,\lambda_1,\cdots\lambda_N\}$$

where $\bar{\bar{B}}^{-1}$ is the inverse of $\bar{\bar{B}}$, $\bar{\bar{B}}^T$ is the transpose of $\bar{\bar{B}}$, diag $\{\cdots\}$ is a diagonal matrix, and λ_0, λ_1,... and λ_N are the characteristic roots of $\bar{\bar{Q}}$.

3) The length of $\bar{X}$ is preserved under the transformation

$$\bar{Y} = \bar{\bar{B}}^{-1}\bar{X} = \bar{\bar{B}}^T\bar{X}$$

or

$$\bar{X} = \bar{\bar{B}}\bar{Y} \tag{32}$$

in the sense that

$$\bar{X}^T\bar{X} = (\bar{\bar{B}}\bar{Y})^T\bar{\bar{B}}\bar{Y} = \bar{Y}^T\bar{Y} \tag{33a}$$

or

$$\sum_{n=0}^{N} X_m^2 = \sum_{m=0}^{N} Y_m^2 \tag{33b}$$

4) The orthogonal matrix can be normalized such that the determinant

$$|B_{ij}| = 1 \tag{34}$$

Using these properties, we can deduce the following results:

1)

$$\sum_{m=0}^{N}\sum_{n=0}^{N} Q_{mn}X_mX_n = \sum_{m=0}^{N}\lambda_m(t_1,t_2)Y_m^2 \tag{35a}$$

where

$$\bar{Y} = \bar{\bar{B}}^T\bar{X} \tag{35b}$$

2)

$$\sum_{m=0}^{N} P_mX_m = \sum_{m=0}^{N}\alpha_m(t_1,t_2)Y_m \tag{36a}$$

where

$$\bar{\alpha}(t_1,t_2) = \bar{\bar{B}}^T\bar{P} \tag{36b}$$

3)

$$dx_0dx_1\ldots dx_N = dy_0dy_1\ldots dy_N \tag{37}$$

from the fact that the Jacobian

$$\left|\frac{\partial(x_0,x_1,\ldots x_N)}{\partial(y_0,y_1,\ldots y_N)}\right| = \begin{vmatrix} \frac{\partial x_0}{\partial y_0} & \frac{\partial x_0}{\partial y_1} & \cdots & \frac{\partial x_0}{\partial y_N} \\ \frac{\partial x_1}{\partial y_0} & \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \vdots & & & \\ \frac{\partial x_N}{\partial y_0} & \frac{\partial x_N}{\partial y_1} & \cdots & \frac{\partial x_N}{\partial y_N} \end{vmatrix} = 1$$

(Since $\bar{X} = \bar{\bar{B}}\bar{Y}$, by the indicial notation $X_i = B_{ij}Y_j$, it follows that $\partial x_i/\partial y_j = B_{ij}$. By Eq. (34), we have at once $|\partial(x_0, x_1,\ldots x_N)/\partial(y_0,y_1,\ldots,y_N)| = 1$.)

With Eqs. (27), (33), and (35-37), we recast Eq. (29) as

$$\phi_{Y\dot{Y}}(t_1,t_2) = \int\cdots\int\exp\left\{i\left[\sum_{m=0}^{N}\alpha_m(t_1,t_2)y_m + \sum_{m=0}^{N}\lambda_m(t_1,t_2)y_m^2 - (\sigma_1t_1+\bar{\sigma}_1t_2)\right]\right\}\cdot(2\pi)^{-(N+1)/2}$$

$$\times\exp\left\{-\frac{1}{2}\sum_{m=0}^{N}y_m^2\right\}dy_0dy_1\ldots dy_N$$

$$= (2\pi)^{-(N+1)/2}\exp[-i(\sigma_1t_1+\bar{\sigma}_1t_2)]$$

$$\cdot\prod_{m=0}^{N}\int\exp[i\alpha_m(t_1,t_2)y_m + i\lambda_m(t_1,t_2)y_m^2 - \tfrac{1}{2}y_m^2]dy_m$$

The final result can be put into a remarkably compact expression:

$$\phi_{Y\dot{Y}}(t_1,t_2) = \left\{\prod_{m=0}^{N}[1-i2\lambda_m(t_1,t_2)]^{-1/2}\right\}$$

$$\cdot\exp\left\{-\frac{1}{2}\sum_{m=0}^{N}\frac{\alpha_m^2(t_1,t_2)}{1-i2\lambda_m(t_1,t_2)} - i(\sigma_1t_1+\bar{\sigma}_1t)\right\} \tag{38}$$

The one-dimensional probability density function can be obtained from the one-dimensional characteristic function, e.g.,

$$p(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_Y(t_1)e^{-it_1y}dt_1 \tag{39}$$

where $\phi_Y(t_1)$ is the one-dimensional characteristic function and is the degenerate case of Eq. (38) by setting $t_2 = 0$. In fact, the explicit result can be rewritten even more simply as

$$\phi_Y(t_1) = \left\{\prod_{m=0}^{N}[1-i2\lambda'_mt_1]^{-1/2}\right\}$$

$$\cdot\exp\left\{-\frac{t_1^2}{2}\sum_{m=0}^{N}\frac{\alpha'_m}{1-2i\lambda'_mt_1} - i\sigma_1t_1\right\} \tag{40}$$

where λ'_m and α'_m are constants, λ'_m being the characteristic roots of $\{A_{mn}\}$ and $\alpha'_m = B_{mn}D_n$.

III. Application to the Gust-Loading Problem

To illustrate the method and demonstrate the effects of the non-Gaussian nature of the atmospheric turbulence on the statistical characteristics of the response, we consider as an example the plunging motion of a rigid aircraft induced by turbulent gust. The turbulence is assumed to be nearly Gaussian, but no such restriction is made on the response.

Equation of Motion

In the case of the plunging motion of a rigid aircraft, the one-degree-of-freedom dynamic model used by many other researchers[1,7] is typically the following:

$$\frac{dV(t)}{dt} + \lambda V(t) = \lambda W(t) \tag{41}$$

In Eq. (41), $V(t)$ is the plunging velocity of the aircraft induced by the turbulent gust velocity $W(t)$, and

$$\lambda = \rho US\frac{dC_L}{d\alpha}/2m \tag{42}$$

where ρ is the air density, S the wing area, U the level flight velocity, m the mass of the aircraft, and $\mathrm{d}C_L/\mathrm{d}\alpha$ the slope of the lift coefficient curve for the aircraft. Clearly, λ^{-1} is the characteristic time of the system. The solution is sought subject to the initial condition

$$V=0, \quad \text{at} \quad t=t_0 \tag{43}$$

where t_0 is the time when the aircraft enters the turbulent region.

We nondimensionalize $V(t)$ and $W(t)$ by choosing the rms of $W(t)$,

$$\sigma_w \equiv \langle W(t)^2 \rangle^{1/2} \tag{44}$$

as the velocity scale, and nondimensionalize t by $2L_w/U$ where $L=2\ L_w$ is the longitudinal length scale of the turbulence.[15] In nondimensionalized quantities, Eq. (41) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\bar{V}+\bar{\lambda}\bar{V}=\bar{\lambda}\bar{W} \tag{45}$$

and Eq. (43) becomes

$$\bar{V}=0, \quad \text{at} \quad \bar{t}=\bar{t}_0 \tag{46}$$

where $\bar{V}=V/\sigma_w$, $\bar{W}=W/\sigma_w$, $\bar{t}=t/(2L_w/U)$, $\bar{t}_0=t_0/(2L_w/U)$, and

$$\bar{\lambda}=\lambda/(U/2L_w) \tag{47}$$

Turbulence Model

Following Wang and Shu[6] and Dutton,[7] we express $\bar{W}(\bar{t})$ by a two-term functional expansion

$$\bar{W}(\bar{t})=\int_{-\infty}^{\infty} P^{(1)}(\bar{t}-\tau)H^{(1)}(\tau)\mathrm{d}\tau + \epsilon\int_{-\infty}^{\infty}\int P^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2)H^{(2)}(\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \tag{48}$$

where $P^{(1)}(\bar{t}-\tau)$ and $P^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2)$ are deterministic functions to be determined from given correlation functions of $\bar{W}(\bar{t})$; ϵ is a small parameter introduced here to indicate the relative order of magnitude of the two terms. Again, if $\epsilon=0$, $\bar{W}(t)$ represented by Eq. (48) becomes a stationary Gaussian process.

We assume now that $\bar{W}(\bar{t})$ is characterized by its two- and three-time correlation functions, defined as

$$\bar{R}_1(\bar{\theta})=\langle \bar{W}(\bar{t})\,\bar{W}(\bar{t}+\bar{\theta})\rangle \tag{49}$$

$$\bar{R}_2(\bar{\theta}_1,\bar{\theta}_2)=\langle \bar{W}(t)\,\bar{W}(\bar{t}+\bar{\theta}_1)\,\bar{W}(\bar{t}+\bar{\theta}_2)\rangle \tag{50}$$

where $\bar{\theta}$, $\bar{\theta}_1$, and $\bar{\theta}_2$ are time shift on the nondimensionalized time history basis. Substituting Eq. (48) into the right-hand side of Eqs. (49) and (50) and using the statistic properties of $H^{(1)}(\tau)$ and $H^{(2)}(\tau_1,\tau_2)$, we obtain

$$\bar{R}_1(\bar{\theta})=\int P^{(1)}(\eta+\bar{\theta})P^{(1)}(\eta)\mathrm{d}\eta + 2\epsilon^2\iint P^{(2)}(\eta_1+\bar{\theta},\eta_2+\bar{\theta})P^{(2)}(\eta_1,\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2$$

$$\begin{aligned}\bar{R}_2(\bar{\theta}_1,\bar{\theta}_2)=&2\epsilon[\iint P^{(1)}(\eta_1)P^{(1)}(\eta_2)P^{(2)}(\bar{\theta}_1+\bar{\theta}_2+\eta_1,\bar{\theta}_2+\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2+\iint P^{(1)}(\eta_1)P^{(1)}(\eta_2)P^{(2)}(\bar{\theta}_1\\&+\eta_1,-\bar{\theta}_2+\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2+\iint P^{(1)}(\eta_1)P^{(1)}(\eta_2)P^{(2)}(-\bar{\theta}_1+\eta_1,\\&-\bar{\theta}_1-\theta_2+\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2]+8\epsilon^3\iiint P^{(2)}(\eta_1,\eta_2)P^{(2)}(\bar{\theta}_1\\&+\eta_2,\eta_3)P^{(2)}(\bar{\theta}_1+\bar{\theta}_2+\eta_2,\bar{\theta}_1+\eta_3)\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\end{aligned} \tag{52}$$

For given $\bar{R}_1(\bar{\theta})$, and $\bar{R}_2(\bar{\theta}_1,\bar{\theta}_2)$, Eqs. (51) and (52) are two simultaneous equations to be solved for $P^{(1)}$ and $P^{(2)}$.

In analyses of this kind, the double correlation of $\bar{W}(\bar{t})$ is often taken to be Dryden's spectrum,[14,15] which yields, by Fourier inversion,

$$R_1(\theta)=\sigma_w^2\left(1-\frac{|\theta|}{4T_w}\right)e^{-|\theta|/2T_w} \tag{53}$$

where $|\theta|$ stands for the absolute value of θ, T_w is the "time scale" of the turbulence, and θ is the time shift on a time history basis. By Taylor's hypothesis, the time scale and length scale are related through the following relations:

$$T_w=2L_w/U; \quad \theta=r/U \tag{54}$$

where r is the space shift in the flight (flow) direction. In terms of the nondimensionalized variables, Eq. (53) becomes

$$\bar{R}_1(\bar{\theta})=\left(1-\frac{|\bar{\theta}|}{2}\right)e^{-|\bar{\theta}|} \tag{55}$$

where $\bar{R}_1=R_1/\sigma_w^2$, $\bar{\theta}=\theta/2T_w$.

We have been unable to find, in the literature, a proper expression for the triple correlation function $\langle \bar{W}(\bar{t})\,\bar{W}(\bar{t}+\bar{\theta}_1)\,\bar{W}(\bar{t}+\bar{\theta}_2)\rangle$ for atmospheric turbulence. As a consequence of Taylor's hypothesis, this quantity in isotropic turbulence is identically zero by the condition of isotropy. In order to make specific calculations, we set in the following demonstration

$$\bar{R}_2(\bar{\theta}_1,\bar{\theta}_2)=0 \tag{56}$$

With the right-hand sides of Eqs. (51) and (52) given, the presence of a small parameter ϵ suggests an iteration procedure of solving this set of equations. The iteration starts from the zeroth order

$$\bar{R}_1(\bar{\theta})=\int_0^{\infty}P^{(1)}(\bar{\theta}+\eta)P^{(1)}(\eta)\mathrm{d}\eta \tag{57}$$

The solution of this nonlinear integral equation, due to Wiener,[16] is expressible in terms of Hilbert integrals. For the special case of Eqs. (55) and (56), the final result neglecting terms of $0(\epsilon^3)$ is found to be

$$P^{(1)}(\eta)=P_0^{(1)}(\eta)+\epsilon^2P_2^{(1)}(\eta) \tag{58}$$

where

$$P_0^{(1)}(\eta)=\begin{cases}\sqrt{3}[1-(1-\beta_1)\eta]e^{-\eta} & \eta\geq 0\\ 0 & \eta<0\end{cases}$$

$$P_2^{(1)}(\eta)=\begin{cases}(\gamma_1+\gamma_2)e^{-\eta}+\gamma_3e^{-2\beta_1\eta} & \eta\geq 0\\ 0 & \eta<0\end{cases}$$

with

$$\gamma_1=C_{22}-C_{23}-C_{21}(1+4\beta_1-2\beta_1^2)/3\beta_1$$

$$\gamma_2=2[C_{22}-C_{21}-\gamma_1(1+\beta_1)/2]/\beta_1$$

$$\gamma_3=C_{21}(1+2\beta_1)^2/3\beta_1$$

$$C_{21}=-C_2^2\left[\frac{a^2}{4}-\frac{a^2}{1+2\beta_1}-\frac{ab}{(1+2\beta_1)^2}\right]$$

$$C_{22}=-C_2^2\left[\frac{a^2}{2}+\frac{ab}{2}+\frac{b^2}{4}-\frac{a^2}{1+2\beta_1}-\frac{ab}{(1+2\beta_1)^2}\right]$$

$$C_{23}=-C_2^2\left[\frac{ab}{2}+\frac{b^2}{4}-\frac{ab}{1+2\beta_1}\right]$$

$$a=\frac{2\beta_1}{(1-2\beta_1)^2},\quad b=\frac{1}{1-2\beta_1},\quad \beta_1=\frac{1}{\sqrt{3}}$$

and

$$P^{(2)}(\eta_1,\eta_2)=\begin{cases} C_2\{2\beta_1-[\eta_2(1-2\beta_1)+2\beta_1]e^{-(1-2\beta_1)\eta_2}\}e^{-\beta_1\eta_1-\beta_1\eta_2}, & \eta_1\geq\eta_2\geq 0\\ C_2\{2\beta_1-[\eta_1(1-2\beta_1)+2\beta_1]e^{-(1-2\beta_1)\eta_1}\}e^{-\beta_1\eta_1-\beta_1\eta_2}, & \eta_2\geq\eta_1\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{59}$$

where C_2 is a normalization factor, found to be 3.2744, such that

$$\iint[P^{(2)}(\eta_1,\eta_2)]^2\mathrm{d}\eta_1\mathrm{d}\eta_2=1$$

For an arbitrary $R_2\neq 0$, we have also constructed the solution which will be presented elsewhere.

Statistical Properties

The turbulence modeling by Eq. (48), via the kernel functions expressed in Eqs. (58) and (59), reproduces the correlation functions R_1 and R_2 of Eqs. (55) and (56). It is now possible to write $\bar{W}(\bar{t})$ as the sum of a Gaussian part, $\bar{W}_G(\bar{t})$, and a non-Gaussian part, $\bar{W}_{NG}(\bar{t})$:

$$\bar{W}(\bar{t})=\bar{W}_G(\bar{t})+\bar{W}_{NG}(\bar{t}) \tag{60}$$

where

$$\bar{W}_G(\bar{t})=\int_{-\infty}^{\infty}[P_0^{(1)}(\bar{t}-\tau)+\epsilon^2P_1^{(1)}(\bar{t}-\tau)]H^{(1)}(\tau)\mathrm{d}\tau \tag{61}$$

and

$$\bar{W}_{NG}(\bar{t})=\epsilon\int_{-\infty}^{\infty}\int P^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2)H^{(2)}(\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \tag{62}$$

Since $H^{(1)}(\tau)$ and $H^{(2)}(\tau_1,\tau_2)$ are statistically orthogonal, it follows from Eq.(60),

$$\langle\bar{W}(\bar{t})^2\rangle=\langle\bar{W}_G(\bar{t})^2\rangle+\langle\bar{W}_{NG}(\bar{t})^2\rangle \tag{63}$$

Let us define

$$R_w^2=\langle\bar{W}_{NG}(\bar{t})^2\rangle/\langle\bar{W}_G(\bar{t})^2\rangle \tag{64}$$

as the ratio of the kinetic energies of the non-Gaussian and the Gaussian components. Using Eq. (61) with the term of order ϵ^4 neglected, we get

$$\langle\bar{W}_G(\bar{t})^2\rangle=1-2\epsilon^2 \tag{65}$$

Equation (62) leads to

$$\langle\bar{W}_{NG}(\bar{t})^2\rangle=2\epsilon^2 \tag{66}$$

Equations (63), (65), and (66) thus satisfy the normalizing condition

$$\langle\bar{W}(\bar{t})^2\rangle=1 \tag{67}$$

From Eqs. (65) and (66), there is also

$$R_w^2=2\epsilon^2/(1-2\epsilon^2) \tag{68}$$

or

$$\epsilon^2=R_w^2/2(1+R_w^2) \tag{69}$$

Reeves et al.[17] analyzed the LO-LOCAT phase III data and reported that the typical atmospheric turbulence has a flatness factor, $\langle W(t)^4\rangle/\langle W(t)^2\rangle^2$, of 3.5. Assuming $R_w^2=0.07$ in our model, it is calculated that $\epsilon^2=0.033$ and the flatness factor = 3.5082. We therefore adopt $R_w^2=0.07$ and evaluate the probability density function using the method presented in Section II. The result is shown in Fig. 1. From their analysis of measured data, Blackadar et al.[19] reported that the probability density functions are larger at the origin than a Gaussian function and become less than the Gaussian function for $|w/\sigma_w|$ in the ranges between 1 and 3. It is seen that Fig. 1 has indeed such features. Figure 2 shows the cumulative probability distributions of the present model of atmospheric turbulence and of the LO-LOCAT phase III data. The agreement is almost perfect.

Note that the only assumptions made in our turbulence modeling are: 1) the non-Gaussian part is small enough to permit a two-term expansion, and 2) $\bar{R}_1$ and $\bar{R}_2$ satisfy Eqs. (55) and (56). Unlike other attempts, no specific mechanism needs to be implied or postulated for the turbulence.

Response of the Aircraft

We write the expansion of the response process $\bar{V}(\bar{t})$ by

$$\bar{V}(\bar{t})=\sum_{n=1}^{\infty}\int\ldots\int K^{(n)}(\bar{t};\tau_1,\tau_2,\ldots\tau_n)\times H^{(n)}(\tau_1,\tau_2,\ldots\tau_n)\mathrm{d}\tau_1\mathrm{d}\tau_2\ldots\mathrm{d}\tau_n \tag{70}$$

Here, $K^{(n)}(\bar{t};\tau_1,\tau_2,\ldots\tau_n)$ is taken to be a function of $\bar{t}$ and $\tau_1,\tau_2,\ldots\tau_n$ separately because, in the transient state, $\bar{V}(\bar{t})$ is not stationary. It also should be noted that the complete expansion is used for $\bar{V}(\bar{t})$, which is not necessarily nearly Gaussian.

We substitute Eq. (70) into Eq. (41). Because of the orthogonal properties of the functionals, the following set of differential equations for the kernel functions of the response are derived

$$\frac{\partial}{\partial\bar{t}}K^{(1)}(\bar{t};\tau)+\bar{\lambda}K^{(1)}(\bar{t};\tau)=\bar{\lambda}P^{(1)}(\bar{t}-\tau) \tag{71}$$

$$\frac{\partial}{\partial\bar{t}}K^{(2)}(\bar{t};\tau_1,\tau_2)+\bar{\lambda}K^{(2)}(\bar{t};\tau_1,\tau_2)=\epsilon\bar{\lambda}P^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2) \tag{72}$$

$$\frac{\partial}{\partial\bar{t}}K^{(3)}(\bar{t};\tau_1,\tau_2,\tau_3)+\bar{\lambda}K^{(3)}(\bar{t};\tau_1,\tau_2,\tau_3)=0 \tag{73}$$

and so forth. We have designated $\bar{t}_0$ as the time the aircraft enters the turbulent field, and assume that initially the aircraft is in horizontal flight; that is,

$$K^{(1)}(\bar{t}_0;\tau)=K^{(2)}(\bar{t}_0;\tau_1,\tau_2)=\ldots=0 \tag{74}$$

for all $\tau_1,\tau_2,\ldots$.

The solutions for the kernels subject to the initial conditions given by Eq. (74) are

$$K^{(1)}(\bar{t};\tau)=\bar{\lambda}\int_{\bar{t}_0}^{\bar{t}}P^{(1)}(\xi-\tau)e^{-\bar{\lambda}(\bar{t}-\xi)}\mathrm{d}\xi \tag{75}$$

$$K^{(2)}(\bar{t};\tau_1,\tau_2)=\epsilon\bar{\lambda}\int_{\bar{t}_0}^{\bar{t}}P^{(2)}(\xi-\tau_1,\xi-\tau_2)e^{-\bar{\lambda}(\bar{t}-\xi)}\mathrm{d}\xi \tag{76}$$

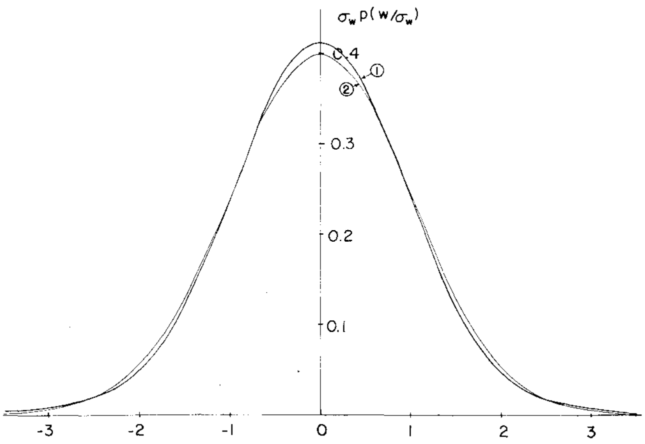

Fig. 1 Comparison of the probability density function of the presently modeled atmospheric turbulence (flatness factor = 3.5082) with a Gaussian density function.

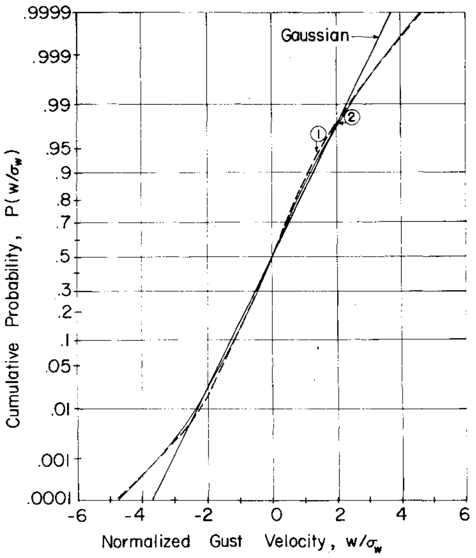

Fig. 2 Cumulative probability distribution of atmospheric turbulence; curve 1-LO-LOCAT phase III data, curve 2- present model.

and

$$K^{(3)}(\bar{t};\tau_1,\tau_2,\tau_3) = K^{(4)}(\ldots) = \ldots = 0 \tag{77}$$

To obtain the stationary solution, we let $\bar{t}_0$ tend to $-\infty$,

$$K^{(1)}(\bar{t};\tau) = \bar{\lambda}\int_{-\infty}^{\bar{t}} P^{(1)}(\xi-\tau)e^{-\bar{\lambda}(\bar{t}-\xi)}d\xi \tag{78}$$

and

$$K^{(2)}(\bar{t};\tau_1,\tau_2) = \epsilon\bar{\lambda}\int_{-\infty}^{\bar{t}} P^{(2)}(\xi-\tau_1,\xi-\tau_2)e^{-\bar{\lambda}(\bar{t}-\xi)}d\xi \tag{79}$$

By changing the variable of integration, Eqs. (78) and (79) lead to

$$K^{(1)}(\bar{t};\tau) = \bar{\lambda}\int_0^{\infty} P^{(1)}(\bar{t}-\tau-\eta)e^{-\bar{\lambda}\eta}d\eta = K^{(1)}(\bar{t}-\tau) \tag{80}$$

and

$$K^{(2)}(\bar{t};\tau_1,\tau_2) = \epsilon\bar{\lambda}\int_0^{\infty} P^{(2)}(\bar{t}-\tau_1-\eta,\bar{t}-\tau_2-\eta)e^{-\bar{\lambda}\eta}d\eta = K^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2) \tag{81}$$

Consequently, the exact solution for the response process $\bar{V}(\bar{t})$ is obtained as

$$\bar{V}(\bar{t}) = \int_{-\infty}^{\infty} K^{(1)}(\bar{t}-\tau)H^{(1)}(\tau)d\tau + \int_{-\infty}^{\infty}\int K^{(2)}(\bar{t}-\tau_1,\bar{t}-\tau_2)H^{(2)}(\tau_1,\tau_2)d\tau_1 d\tau_2 \tag{82}$$

The terms $K^{(1}$ and $K^{(2)}$ are provided by substituting Eqs. (58) and (59) into Eqs. (80) and (81), respectively.

The two-time correlation function of $\bar{V}(t)$ can be calculated in two ways. One is by Eq. (82):

$$\langle \bar{V}(\bar{t})\bar{V}(\bar{t}+\bar{\theta})\rangle = \int_{-\infty}^{\infty} K^{(1)}(\bar{\theta}+\xi)K^{(1)}(\xi)d\xi + 2\int_{-\infty}^{\infty}\int K^{(2)}(\bar{\theta}+\xi_1,\bar{\theta}+\xi_2)K^{(2)}(\xi_1,\xi_2)d\xi_1 d\xi_2 \tag{83}$$

Another is by the classical spectrum approach:

$$\langle \bar{V}(\bar{t})\bar{V}(\bar{t}+\bar{\theta})\rangle = \bar{\lambda}^2\int_0^{\infty}\int\langle \bar{W}(\bar{t}-\xi_1)\bar{W}(\bar{t}+\bar{\theta}-\xi_2)\rangle e^{-\bar{\lambda}(\xi_1+\xi_2)}d\xi_1 d\xi_2 \tag{84}$$

It has been proven that the two results are identical. The joint probability density functionof $\bar{V}(\bar{t})$ and $\bar{V}(t)$ can becomputed using the method presented in Section II.

To demonstrate the effects of the non-Gaussian nature of the atmospheric turbulence on the response statistical characteristics, we have computed the one-dimensional probability density functions for the case $R_w^2 = 0.07$, with $\bar{\lambda} = 0.125$, 0.25, 0.5, 1.50, and 5.0. In Fig. 3 the cumulative distributions are shown. In Fig. 4, we present the response flatness factor calculated by using Eq. (82). As a confirmation, the flatness factors for the case $R_w^2 = 0.07$ with $\bar{\lambda} = 0.125$, 0.25, 0.5, 1.5, and 5.0 are also calculated by using the probability density function obtained. The results calculated by these two methos are almost indistinguishable from each other.

IV. Conclusions

The expansion of a random function in terms of statistically orthogonal random functions is a logical extension of the usual expansion techniques for deterministic functions. While conceptually straightforward, it does not seem to have made much headway in aeromechanical engineering applications—no doubt because the procedure and its details to arrive at results of practical interest remain largely unknown. The emphasis of the present study is twofold: to explore the feasibility aspects of the method, at least as applied to a simple system, and to demonstrate the richness of the information that can be deduced from the result.

For a stationary random function, once the expansion kernel functions are known, a method to construct the probability density functions is presented. To obtain the expansion kernel functions, we follow the integral equation formulation in terms of the correlation functions which are assumed given. At least for slightly non-Gaussian random functions, the set of simultaneous nonlinear integral equations can be attacked by an iterative procedure.

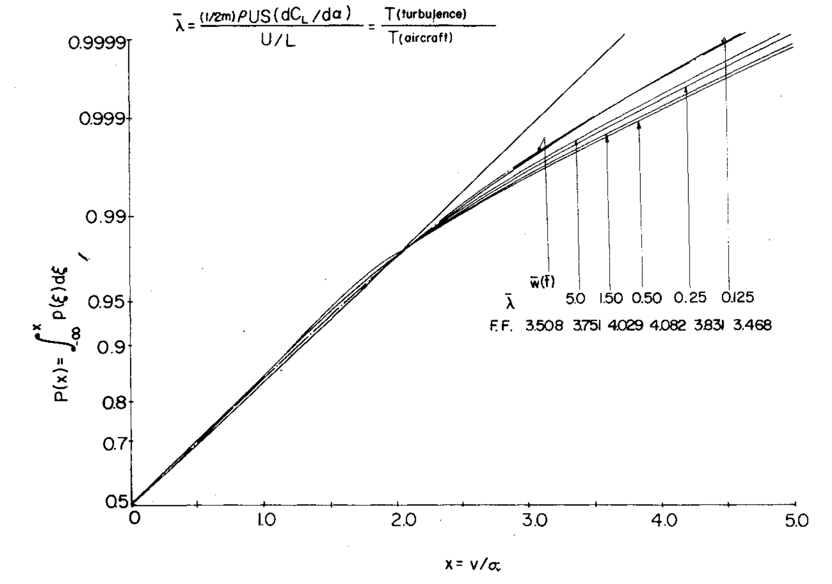

Fig. 3 Response distribution with $\bar{\lambda}$ as parameter.

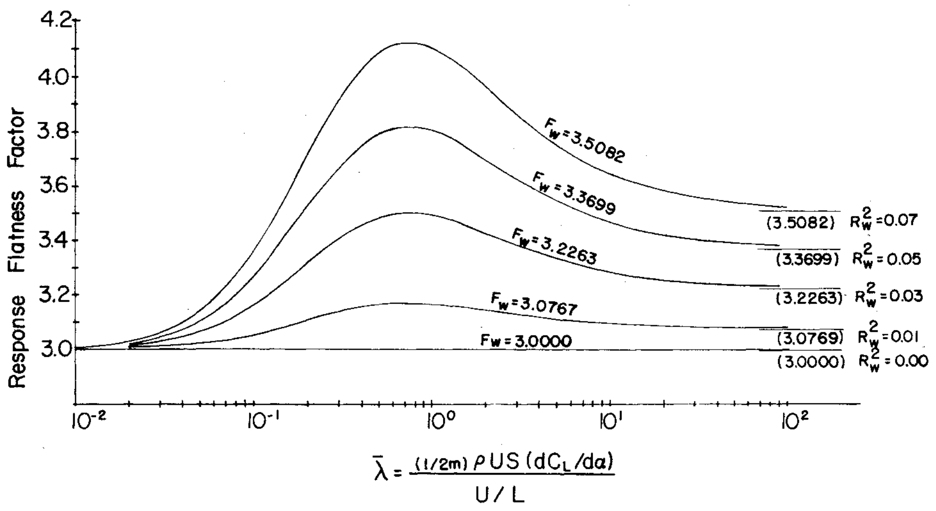

Fig. 4 Response flatness factor where F_w is the turbulence flatness factor.

In application to the gust loading problem, a two-term expansion of the atmospheric turbulence is first made to yield the typical flatness factor of 3.5 and to produce the Dryden spectrum. The probability density function so constructed appears to be in excellent agreement with available measurements. This is unlikely to be coincidental because the smallness parameter of the method to justify a two-term expansion is indeed small.

Without restricting to slightly non-Gaussian functions, we find the response of the aircraft as a first-order linear system to be representable also by a two-term expansion. The probability density function $p(v)$ for the response velocity $V(t)$ and the joint probability density function $p(v,\dot{v})$ for velocity $V(t)$ and acceleration $\dot{V}(t)$ can thus be constructed using the technique presented in Section II. The cumulative probability density function for $V(t)$ is plotted in Fig. 3. The flatness factor of the response velocity $\langle V^4 \rangle / \langle V^2 \rangle^2$ is plotted in Fig. 4. Those curves depend on the parameter $\bar{\lambda}$, the ratio of the characteristic time of the aircraft and that of the turbulence. In most practical cases, the value of this ratio is said to lie between 1.25 and 5.0.[1] It may be concluded from Figs. 3 and 4 that the frequency of occurrence of large gust velocity $W(t)$ is higher than that estimated by a Gaussian assumption, and that of the occurrence of large response velocity $V(t)$ is even higher. Based on the joint probability density fucntion, other statistical properties such as the level crossing frequency, of obvious design interest, can also be obtained.

It may be emphasized that our approach based on the expansion technique should have wide applicability to other random response problems subject to a slightly non-Gaussian forcing function. As for the aircraft gust loading problem, the extension to multiple degrees of freedom is immediate, at least in principle, as long as the response can be treated as that of a linear system.

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